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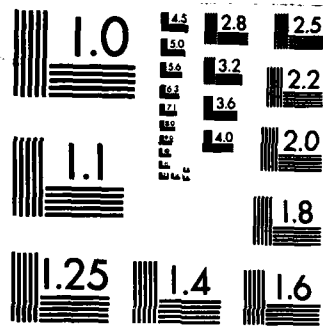MATHEMATICS RESEARCH CENTER   B HARRIS ET AL.   MAR 85
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THE OPTIMALITY OF LOWER CONFIDENCE
LIMITS FOR THE RELIABILITY OF SERIES
SYSTEMS OBTAINED BY THE METHOD OF KEY
TEST RESULTS OR OTHER RELATED
TECHNIQUES

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THE OPTIMALITY OF LOWER CONFIDENCE LIMITS FOR THE RELIABILITY
OF SERIES SYSTEMS OBTAINED BY THE METHOD OF KEY
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ABSTRACT

In order to obtain lower confidence limits for the reliability of series systems using binomial subsystem data, K. A. Weaver introduced the method of "key test results". This work was extended by A. Winterbottom. In the present paper, conditions are obtained under which the "method of key test results" gives Buehler optimal lower confidence limits identical with those given by the ordering induced by the maximum likelihood estimator.

SIGNIFICANCE AND EXPLANATION

In this paper, conditions are established under which the total number of defects may be used with no loss in obtaining the lower confidence limit for the reliability of a series system. In such a case, the computations are reduced to the elementary task of determining a lower confidence limit for a single binomial parameter.

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THE OPTIMALITY OF LOWER CONFIDENCE LIMITS FOR THE RELIABILITY OF SERIES SYSTEMS

OBTAINED BY THE METHOD OF KEY TEST RESULTS OR OTHER RELATED TECHNIQUES

Bernard Harris and Andrew P. Soms

1. <u>Introduction</u>. Let $Y_1, Y_2, \ldots, Y_k$, $k \geq 2$ be independent binomial random variables with parameters (n_i, p_i), $i = 1, 2, \ldots, k$. The objective is to obtain a $1 - \alpha$ lower confidence limit for $h(\tilde{p}) = \prod_{i=1}^{k} p_i$.

This problem arises naturally in reliability theory, where $h(\tilde{p})$ is the reliability of a series systems of independent components: The data is obtained from independent binomial subsystem experiments. Thus $Y_1, Y_2, \ldots, Y_k$ are the number of times each of the k subsystems functioned in $n_1, n_2, \ldots, n_k$ Bernoulli trials respectively. We let $X_i = n_i - Y_i$, $i = 1, 2, \ldots, k$, be the number of times each of the k subsystems failed in the experiment.

Let

$$f(\tilde{x}, \tilde{p}) = \prod_{i=1}^{k} \binom{n_i}{x_i} p_i^{n_i - x_i} (1 - p_i)^{x_i} = P_{\tilde{p}}(\tilde{X} = \tilde{x}) \qquad (1.1)$$

and let

$$g(\tilde{x}) = \prod_{i=1}^{k} (n_i - x_i) \qquad (1.2)$$

and let

$$\hat{h(p)} = g(\tilde{x}) / \prod_{i=1}^{k} n_i . \qquad (1.3)$$

It is easily verified that $\hat{h(p)}$ is the maximum likelihood estimator of $h(\tilde{p})$

and the minimum variance unbiased estimator of $h(\tilde{p})$. For this reason, the performance of lower confidence limits based on $g(\tilde{x})$ provides a yardstick against which the performance of other criteria can be assessed. In particular, we call $m(\tilde{x})$ an ordering function if whenever $\tilde{x}^{(1)} < \tilde{x}^{(2)}$, we have $m(\tilde{x}^{(1)}) > m(\tilde{x}^{(2)})$, $\tilde{x}^{(i)} = (x_1^{(i)},\ldots,x_k^{(i)})$, $0 < x_j^{(i)} < n_i$; $\tilde{x}^{(1)} < \tilde{x}^{(2)}$ means $x_j^{(1)} < x_j^{(2)}$, $j = 1,2,\ldots,k$. It is clear that $g(\tilde{x})$ is an ordering function. Another ordering function of interest to this discussion is $s(\tilde{x}) = \sum_{i=1}^{k} (n_i - x_i)$. In the discussion that follows, it will be convenient to index the subsystems so that $n_1 < n_2 < \cdots < n_k$. This will entail no loss of generality.

For any ordering function $m(\tilde{x})$, the optimal $1 - \alpha$ lower confidence limit determined by $m(\tilde{x})$ is

$$a_{m(\tilde{x}_0)} = \inf\{h(\tilde{p}) \,|\, P_{\tilde{p}}\{m(\tilde{x}) > m(\tilde{x}_0)\} = \alpha\} , \qquad (1.4)$$

where $\tilde{x}_0$ denotes the observed outcome.

An experimental outcome is said to be a "key test result" (A. K. Weaver [2]) if $x_2 = n_2,\ldots,x_k = n_k$. We examine Weaver's analysis of key test results in Section 2. A. Winterbottom's [3] extension of Weaver's work is then discussed. We also discuss some other techniques which depends only on the total number of failures.

In Section 3, we present the main points of the paper. There we obtain conditions under which the optimal lower confidence limit for $h(\tilde{p})$ using an ordering function depending on the total number of failures coincides with that given by $g(\tilde{x})$. Correspondingly, we also obtain conditions under which the two procedures differ. These conclusions are compared with the results obtained by Winterbottom and others.

2. The Method of Key Test Results and Other Techniques Based on the Number of Failures.

Weaver [2] introduced the notion of a "key test result". He studied the analysis of key test results for $n_1 = n_2 = \cdots = n_k = n$. In order to motivate the subject matter of the next section, we summarize his results.

To facilitate obtaining a solution, Weaver replaced (1.4) by

$$a'_{s(\tilde{x}_0)} = \inf_{\tilde{p} \in P^*} \{h(\tilde{p}) \mid P_{\tilde{p}}\{s(\tilde{X}) > s(\tilde{x}_0)\} = \alpha\} , \qquad (2.1)$$

where

$$P^* = \{\tilde{p} : p_1 = p_2 = \cdots = p_j = \rho, \ p_{j+1} = p_{j+2} = \cdots = p_k = 1, \ 1 < j < k\} .$$

Then

$$P_{\tilde{p}}\{s(\tilde{X}) > s(\tilde{x}_0)\} = \sum \prod_{i=1}^{j} \binom{n}{x_i} \rho^{n-x_i}(1-\rho)^{x_i} , \qquad (2.2)$$

where the sum is over all $x_1, x_2, \ldots, x_j$ such that $kn - \sum_{i=1}^{j} x_i > s(\tilde{x}_0)$.

For $\tilde{p} \in P^*$, $h(\tilde{p}) = \rho^j$. In addition, since $p_1 = p_2 = \cdots = p_j = \rho$, we can write

$$P\{\sum_{i=1}^{k} X_i = z\} = \binom{jn}{z}\rho^{jn-z}(1-\rho)^z$$

and thus

$$P\{kn - \sum_{i=1}^{j} X_i > s(\tilde{x}_0)\} = P\{\sum_{i=1}^{k} X_i < kn - s(\tilde{x}_0)\}$$

$$= \sum_{z=0}^{kn-s(\tilde{x}_0)} \binom{jn}{z}\rho^{jn-z}(1-\rho)^z . \qquad (2.3)$$

For fixed j, the equation

$$f(j,\rho) = \sum_{z=0}^{kn-s(\tilde{x}_0)} \binom{jn}{z}\rho^{jn-z}(1-\rho)^z = \alpha, \qquad 0 < \alpha < 1 ,$$

has a unique solution in ρ, which we denote by $\rho_j(\alpha)$. Thus, to obtain the infimum required in (2.1), it suffices to set $\rho = \rho_j(\alpha)$ in (2.3), then

$$a'_{s(\widetilde{x}_0)} = \min\{\rho_1(\alpha),(\rho_2(\alpha))^2,\ldots,(\rho_k(\alpha))^k\} \ . \tag{2.4}$$

Weaver gave some heuristic arguments which suggest that for $\rho_j(\alpha)$ sufficiently close to unity for all j, $a'_{s(\widetilde{x}_0)} = \rho_1(\alpha)$. This conclusion appears to hold with much greater generality than required by Weaver. This impression is based on extensive numerical experimentation. If that conclusion is assumed valid, then Weaver's method is an extraordinarily simple method of obtaining lower confidence limits for the reliability of series systems of independent subsystems. Consequently, it seems desirable to investigate its properties.

Weaver had a number of additional suggestions, but without proofs. In actuality he proposed a more complicated ordering function than $s(\widetilde{x})$, but he did not make substantive use of this ordering function. For experimental outcomes more general than key test results, he proposed comparing

$$\left\{ \sum_{0}^{s(x)-1} \binom{n}{x}\rho_1^{n-x}(1 - \rho_1)^x, \ \sum_{0}^{s(x)} \binom{jn}{x}(\rho_j)^{jn-x}(1 - \rho_j)^x, \ j = 2,3,\ldots,k \right\} \ .$$

Each of the k expressions above are equated to α and the minimum of $\rho_1(\alpha),\rho_2^2(\alpha),\ldots,\rho_k^k(\alpha)$ is taken as the lower confidence limit. Note that $\rho_1(\alpha)$ uses $s(x) - 1$ as the upper level of summation. For unequal sample sizes, Weaver suggested replacing each x_i by $n_1 x_i/n_i$. Weaver notes that this should provide a conservative result. (Intuitively this would seem to be the case).

The use of $s(\widetilde{x})$ as an ordering function has a long history. It is suggested by the following considerations. From (1.3)

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$$h(\hat{p}) = \prod_{i=1}^{k} (n_i - X_i) \Big/ \prod_{i=1}^{k} n_i .$$

Assume that $p_1, p_2, \ldots, p_k$ are all "close to unity", so that the X_i, $i = 1, 2, \ldots, k$ assume "small values" with probability "close to unity". Also assume that $n_1, n_2, \ldots, n_k$ are all large. Then, approximate as follows:

$$h(\hat{p}) = \prod_{i=1}^{k} (1 - \frac{X_i}{n_i}) \sim 1 - \sum_{i=1}^{k} \frac{X_i}{n_i} . \qquad (2.5)$$

If the n_i, $i = 1, 2, \ldots, k$ are "approximately equal", one replaces n_i by an average value, say $\bar{n}$. Thus,

$$h(\hat{p}) \sim 1 - \sum_{i=1}^{k} X_i / \bar{n} \qquad (2.6)$$

and the statistic $h(\hat{p})$ depends only on $\sum_{i=1}^{k} X_i$ and thus is statistically equivalent to $s(\tilde{X})$.

Methods using the approximations (2.5) and (2.6) are discussed in I. V. Pavlov [1].

Winterbottom [3] studied the use of key test results. He proposed basing the lower confidence limit on the key test results $([n_1 h(\hat{p})], n_2, \ldots, n_k)$, $([n_1 h(\hat{p})] + 1, n_2, \ldots, n_k)$, $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$. However, in the theoretical appendix to the paper, he studies conditions under which key test results provide the same lower confidence limit as the optimal lower confidence limit based on $h(\hat{p})$. In this instance, he assumes $n_1 = n_2 = \cdots = n_k = n$ and utilizes the condition that $\sum_{i=1}^{k} x_i \leqslant x_{1,0}$, where $x_{1,0}$ is the observed number of failures on the first component (since this is a key test result).

Specifically, Winterbottom concludes that this holds whenever $(kn - \sum_{i=1}^{k} x_i) > s^*$, where

$$s^* = \max_{0 < s < n} \{s \mid k[n^{k-1}(s - 1)]^{1/k} < (k - 1)n + s - 2\} . \qquad (2.7)$$

If $n_k > n_1$, for key test results, Winterbottom uses the same lower limit that would be obtained if $n_k = n_1$ and asserts that if the lower confidence limit obtained for key test results satisfies (2.7), then this lower confidence limit will also agree with that given by the optimal confidence limit determined by $\hat{h}(p)$.

3. **Comparison of Methods Based on the Number of Failures and Methods Based on the Minimum Variance Unbiased Estimator.** In this section, we obtain the theoretical results which provide the comparison between optimal lower confidence limits based on the two different procedures discussed in the previous sections. To establish the principal result, several preliminary lemmas are introduced.

Lemma 3.1. Let $0 < n_1 \leq n_2 \leq \cdots \leq n_k$, $k \geq 2$ and $0 \leq x_i \leq n_i$, $i = 1,2,\ldots,k$ be given. Then if $\sum_{i=1}^{k} x_i$ is fixed,

$$\prod_{i=1}^{k} (n_i - x_i) \geq (n_1 - \sum_{i=1}^{k} x_i) \prod_{i=2}^{k} n_i . \tag{3.1}$$

Proof. If $\sum_{i=1}^{k} x_i \geq n_1$, the conclusion is obtained trivially. Hence assume $\sum_{i=1}^{k} x_i < n_1$. For $1 < j \leq k$, since $n_j \geq n_1$, we have

$$(n_j - n_1)x_j + x_j \sum_{i=1}^{j-1} x_i \geq 0 . \tag{3.2}$$

Then,

$$x_j \sum_{i=1}^{j-1} x_i + n_1 n_j - n_j \sum_{i=1}^{j-1} x_i - n_1 x_j \geq n_1 n_j - n_j \sum_{i=1}^{j-1} x_i - n_j x_j ,$$

or

$$(n_1 - \sum_{i=1}^{j-1} x_i)(n_j - x_j) \geq n_j(n_1 - \sum_{i=1}^{j} x_i) .$$

Successively setting $j = 2,3,\ldots,k$, we obtain the following inequalities:

$$(n_1 - x_1)(n_2 - x_2) \geqslant n_2(n_1 - x_1 - x_2)$$

$$(n_1 - x_1 - x_2)(n_3 - x_3) \geqslant n_3(n_1 - x_1 - x_2 - x_3)$$

$$\vdots \qquad\qquad \vdots$$

$$(n_1 - x_1 - x_2 - \cdots - x_{k-1})(n_k - x_k) \geqslant n_k(n_1 - x_1 - \cdots - x_k) . \qquad (3.3)$$

The conclusion follows readily upon multiplying the above inequalities.

__Lemma 3.2.__ Let $0 < n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$ and $0 \leqslant x_i < n_i$, $i = 1,2,\ldots,k$ with $\sum\limits_{i=1}^{k} x_i < n_1$ be given. Let $0 \leqslant z_i < n_i$, $i = 1,2,\ldots,k$. Then if

$$A = \{z : \sum_{i=1}^{k} z_i \leqslant \sum_{i=1}^{k} x_i\}, \qquad A^{*} = \{z : \prod_{i=1}^{k} (n_i - z_i) \geqslant \prod_{i=1}^{k} (n_i - x_i)\} ,$$

a necessary condition that $A = A^{*}$ is, $x_i = 0$, $i = 1,2,\ldots,k$, or $x_j = \gamma$, $0 < \gamma < n_1$, $x_i = 0$, $i \neq j$, and $n_j = n_1$.

__Proof.__ Let $B = \{\tilde{z} : \sum\limits_{i=1}^{k} z_i = \sum\limits_{i=1}^{k} x_i\}$. If $\tilde{z} \in B \cap A^{*}$, then, utilizing Lemma 3.1, we have

$$\prod_{i=1}^{k} (n_i - z_i) \geqslant \prod_{i=1}^{k} (n_i - x_i) \geqslant (n_1 - \sum_{i=1}^{k} x_i) \prod_{i=2}^{k} n_i$$

$$= (n_1 - \sum_{i=1}^{k} z_i) \prod_{i=2}^{k} n_i . \qquad (3.4)$$

Let $\tilde{z}^{*} = (\sum\limits_{i=1}^{k} x_i, 0, \ldots, 0)$. Then $\tilde{z}^{*} \in B \subset A$ and since $\sum\limits_{i=1}^{k} x_i < n_1$ was hypothesized, it follows that $0 \leqslant z_i^{*} < n_i$, $i = 1,2,\ldots,k$. Thus,

$$\prod_{i=1}^{k} (n_i - z_i^{*}) = (n_1 - \sum_{i=1}^{k} x_i) \prod_{i=2}^{k} n_i . \qquad (3.5)$$

Assume that $\tilde{z}^{*} \in A^{*}$; then

$$(n_1 - \sum_{i=1}^{k} x_i) \prod_{i=2}^{k} n_i \geqslant \prod_{i=1}^{k} (n_i - x_i) \geqslant (n_1 - \sum_{i=1}^{k} x_i) \prod_{i=2}^{k} n_i . \qquad (3.6)$$

order that (3.6) holds, equality must hold in each of the inequalities (3.3), equivalently in (3.2). Thus,

$$(n_j - n_1)x_j + x_j \sum_{i=1}^{j-1} x_i = 0, \quad j = 2,\ldots,k .\tag{3.7}$$

nce $n_j > n_1$, $0 < x_j$; it follows that

$$(n_j - n_1)x_j = 0, \quad x_j \sum_{i=1}^{j-1} x_i = 0, \quad j = 2,\ldots,k .\tag{3.8}$$

sume that $x_s \neq 0$, $s < k$, then $x_s(x_1 + x_2 + \cdots + x_{s-1}) = 0$, which implies at $x_i = 0$, $i < s$. Furthermore, $(n_s - n_1) x_s = 0$ implies $n_s = n_1$ and us $n_1 = n_2 = \cdots = n_s$.

rollary. If $x_i = 0$, $i = 1,2,\ldots,k$, then $A = A^*$.

roof. If $\tilde{z} \in A$, then $z_i = 0$, $i = 1,2,\ldots,k$ and
$$\prod_{i=1}^{k} (n_i - z_i) = \prod_{i=1}^{k} n_i = \prod_{i=1}^{k} (n_i - x_i). \text{ Thus } \tilde{z} \in A^*. \text{ If } \tilde{z} \in A^*, z_i = 0,$$
$= 1,2,\ldots,k$ and $\tilde{z} \in A$.

The following discussion shows that the condition of Lemma 3.2 is not ufficient. If the condition holds, namely $x_j = \gamma$, $0 < \gamma < n_1$, $x_i = 0$, $i \neq j$, $j = n_1$, $1 < j < k$. Then $\sum_{i=1}^{k} x_i = \gamma$ and $\prod_{i=1}^{k} (n_i - x_i) = (n_1 - \gamma) \prod_{i=2}^{k} n_i$. We onstruct a $\tilde{z} \in A^*$ with $\tilde{z}$ not in A.

Thus let $n_1 < n_2 < \cdots < n_k$ be given with $z_1 = z_2 = \cdots = z_{k-1} = 0$, $_k = \beta > \gamma$. Then

$$\prod_{i=1}^{k} (n_i \quad :_i) = (n_1 - \gamma) \prod_{i=2}^{k} n_i, \quad \prod_{i=1}^{k} (n_i - z_i) = (n_k - \beta) \prod_{i=1}^{k-1} n_i .$$

n order that $\tilde{z} \in A^*$, we must have

$$(n_k - \beta) \prod_{i=1}^{k-1} n_i > (n_1 - \gamma) \prod_{i=2}^{k} n_i$$

or

$$(n_k - \beta)n_1 > (n_1 - \gamma)n_k .$$

This holds whenever

$$\beta n_1 < \gamma n_k .$$

However, $\tilde{z} \notin A$ whenever $\beta > \gamma$. Thus, it suffices to set $\beta = \gamma + 1$. Then there is a $\tilde{z} \in A^*$ with $\tilde{z} \notin A$ whenever

$$\frac{\gamma + 1}{\gamma} < \frac{n_k}{n_1} .$$

In particular, for $\gamma = 1$, $A \neq A^*$ if $n_k > 2n_1$ and if $\gamma = 2$, $A \neq A^*$ whenever $n_k > (3/2)n_1$.

Lemma 3.3. Let $0 < n_1 < n_2 < \cdots < n_k$, $k > 2$, $0 < \gamma < n_1$ be given. Let $\tilde{z} = (z_1, z_2, \ldots, z_k)$, $0 < z_i < n_i$, $i = 1, 2, \ldots, k$. Let $A_\gamma = \{\tilde{z} : \sum_{i=1}^{k} z_i < \gamma\}$, $A_\gamma^* = \{\tilde{z} : \prod_{i=1}^{k} (n_i - z_i) > (n_1 - \gamma) \prod_{i=2}^{k} n_i\}$. Then $A_\gamma \subset A_\gamma^*$, $\alpha = 1, 2, \ldots, n_1 - 1$.

Proof. If $\gamma = 0$, the conclusion is immediate from the Corollary to Lemma 4.2. Thus if $n_1 = 1$, the lemma is established. Hence, assume $n_1 > 1$.

We proceed by induction. Assume $A_\gamma \subset A_\gamma^*$, $\gamma = 0, 1, \ldots, m$, $m < n_1 - 1$. From their respective definitions, it is immediate that

$$A_\gamma \subset A_{\gamma+1}, \quad A_\gamma^* \subset A_{\gamma+1}^* . \tag{3.9}$$

Thus, from the induction hypothesis, it is immediate that

$$A_\gamma \subset A_{\gamma+1}^* . \tag{3.10}$$

Consequently, the conclusion is established if we show that

$$A_{\gamma+1} - A_\gamma \subset A_{\gamma+1}^* .$$

However

$$A_{\gamma+1} - A_\gamma = \{\tilde{z} : \sum_{i=1}^{k} z_i = \gamma + 1\} .$$

From Lemma 3.1; if $\tilde{z} \in A_{\gamma+1} - A_\gamma$, then

$$\prod_{i=1}^{k} (n_i - z_i) > (n_1 - \gamma - 1) \prod_{i=2}^{k} n_i ,$$

consequently such $\tilde{z}$ is in $A_{\gamma+1}^*$, establishing the conclusion.

__Lemma 3.4.__ Let $0 < n_1 < n_2 < \cdots < n_k$, $0 < z_i < n_i$, $i = 1,2,\ldots,k$. Let $A_\gamma^C = \{\tilde{z} : \sum_{i=1}^{k} z_i > \gamma + 1\}$ where z_i, $i = 1,2,\ldots,n$ are real valued and $\gamma < n_1$ is a positive integer.

Let j be the least index, $1 < j < k$ such that

$$n_j - (\sum_{i=j}^{k} n_i - \gamma - 1)/(k - j + 1) > 0 . \tag{3.11}$$

Then

$$\max_{\tilde{z} \in A_\gamma^C} \prod_{i=1}^{k} (n_i - z_i) = \left[\frac{(n_j - \gamma) + \sum_{j+1}^{k} n_i - 1}{k - j + 1} \right]^{k-j+1} \prod_{i=1}^{j-1} n_i . \tag{3.12}$$

__Proof.__ It suffices to consider vectors $\tilde{z}$ such that $\sum_{i=1}^{k} z_i = \gamma + 1$, since, if $\sum_{i=1}^{k} z_i > \gamma + 1$, reducing any positive component of $\tilde{z}$ will increase $\prod_{i=1}^{k} (n_i - z_i)$. Hence, let

$$C = \{\tilde{z} : \sum_{i=1}^{k} z_i = \gamma + 1, \ 0 < z_i < n_i, \ i = 1,2,\ldots,k\}$$

and let

$$D = \{\tilde{z} : z_i < n_i, \ i = 1,2,\ldots,k\} .$$

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Clearly C and D are convex sets. Furthermore, on D, $\sum_{i=1}^{k} \log(n_i - z_i)$ is a strictly concave function. Thus it has a unique maximum on

$D^* = \{\tilde{z} : \sum_{i=1}^{k} z_i = \gamma + 1, z_i < n_i, i = 1, 2, \ldots, k\}$, which we now characterize.

Using Lagrange multipliers or alternatively, from the arithmetic mean – geometric mean inequality, we have

$$\max_{\tilde{z} \in D^*} \prod_{i=1}^{k} (n_i - z_i) = \left\{ \frac{(n_1 - \gamma) + \sum_{i=2}^{k} n_i - 1}{k} \right\}^k = g(\tilde{n}, \gamma, k) . \quad (3.13)$$

The maximum is attained for $\tilde{z}^* = (z_1^*, z_2^*, \ldots, z_k^*)$, where

$$z_i^* = n_i - (\sum_{i=1}^{k} n_i - \gamma - 1)/k, \quad i = 1, 2, \ldots, k , \quad (3.14)$$

$z_1^* < z_2^* < \cdots < z_k^*$.

Thus, since $C \subset D^*$,

$$\max_{\tilde{z} \in C} \prod_{i=1}^{k} (n_i - z_i) \leqslant \max_{\tilde{z} \in D^*} \prod_{i=1}^{k} (n_i - z_i) \quad (3.15)$$

and equality holds whenever $\tilde{z}^*$ is in C. Since $\sum_{i=1}^{k} z_i^* = \gamma + 1 > 0$, there is a least index i such that $z_i^* > 0$. If $\tilde{z}^*$ is in C^C, then $z_1^* < 0$. From the strict concavity of $\sum_{i=1}^{k} \log(n_i - z_i)$ and since $C \subset D^*$, $\max_{\tilde{z} \in C} \prod_{i=1}^{k} (n_i - z_i)$ will be attained on the boundary of C whenever $\tilde{z}^*$ is in C^C. Thus, set $z_1 = 0$ and repeat the above computations with k replaced by k -1, $i = 2, \ldots, k$. That is, set $\tilde{z}_1^* = (0, z_{21}^*, \ldots, z_{k1}^*)$, where

$$z_{i1}^* = n_i - (\sum_{i=2}^{k} n_i - \gamma - 1)/(k - 1), \quad i = 2, \ldots k , \quad (3.16)$$

$$g_1(\tilde{n}, \gamma, k) = n_1 \prod_{i=2}^{k} (n_i - z_{i1}^*) = n_1 \left(\frac{\sum_{i=2}^{k} n_i - \gamma - 1}{k - 1} \right)^{k-1} . \tag{3.17}$$

If $z_{21}^* < 0$, repeat the process as indicated above. This is to be continued until the solution is in C. It can be easily seen that the process terminates, for it $z_1^*, z_{21}^*, \ldots, z_{k-1,k-2}^*$ are all negative, then $\tilde{z}_k^* = (0, 0, \ldots, 0, z_{k,k-1}^*)$ and $z_{k,k-1}^* > 0$ since $\sum_{i=1}^{k} z_i = \gamma + 1$ implies $z_{k,k-1}^* = \gamma + 1 > 0$. The conclusion follows immediately and is given by (3.11) and (3.12).

Lemma 3.5. Let $0 < n_1 < n_2 < \cdots < n_k$, $0 < z_i < n_i$, $i = 1, 2, \ldots, k$. Let $\gamma < n_1$ be a non-negative integer. Let

$$A_\gamma = \{\tilde{z} : \sum_{i=1}^{k} z_i < \gamma\}, \quad A_\gamma^* = \{z : \prod_{i=1}^{k} (n_i - z_i) > (n_1 - \gamma) \prod_{i=2}^{k} n_i\} . \tag{3.18}$$

Then if $A_\gamma^* \subset A_\gamma$, we have $A_{\gamma-1}^* \subset A_{\gamma-1}$.

Proof. If $\gamma = 0$, the lemma is trivially true. Hence assume $\gamma > 0$. Now $A_\gamma^* \subset A_\gamma$ is equivalent to $A_\gamma^c \subset A_\gamma^{*c}$, or whenever $\sum_{i=1}^{k} z_i > \gamma$ it follows that $\prod_{i=1}^{k} (n_i - z_i) < (n_1 - \gamma) \prod_{i=2}^{k} n_i$. Let $\tilde{z} \in A_\gamma^c$ and assume $\tilde{z} \in A_\gamma^{*c}$. Then there is a z_j such that $z_j > 0$. Let $\tilde{z}_1 = (z_1, z_2, \ldots, z_j - 1, \ldots, z_k)$. Then $\tilde{z}_1 \in A_{\gamma-1}^c$ and

$$\prod_{i=1}^{k} (n_i - z_{1i}) = \prod_{i=1}^{k} (n_i - z_i) + \prod_{i \neq j} (n_i - z_i) < (n_1 - \gamma) \prod_{i=2}^{k} n_i + \prod_{i \neq j} n_i .$$

If $j = 1$,

$$(n_1 - \gamma) \prod_{i=2}^{k} n_i + \prod_{i=2}^{k} n_i = (n - \gamma + 1) \prod_{i=2}^{k} n_i$$

and $\tilde{z}_1 \in A_{\gamma-1}^{*c}$.

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Similarly, if $j \neq 1$, then

$$\prod_{i \neq j} n_i < \prod_{i=2}^{k} n_i$$

upon replacing n_1 by n_j; and again $\tilde{z}_1 \in A_{\gamma-1}^*$.

Combining the preceding lemmas, we can establish the following theorems.

<u>Theorem 3.1.</u> Let $0 < n_1 < n_2 < \cdots < n_k$, $k > 2$, $0 < x_i < n_i$, $i = 1,2,\ldots,k$ be given with $\sum_{i=1}^{n} x_i < n_1$. Let $\tilde{z} = (z_1, z_2, \ldots, z_k)$, where $0 < z_i < n_i$, $i = 1,2,\ldots,k$. Let $A = \{\tilde{z} : \sum_{i=1}^{k} z_i < \sum_{i=1}^{k} x_i \}$,

$$A^* = \{z : \prod_{i=1}^{k} (n_i - z_i) > \prod_{i=1}^{k} (n_i - x_i)\} .$$

Then $A = A^*$ if and only if $x_j = \gamma$, $0 < \gamma < n_1$, $x_i = 0$, $i \neq j$, $n_j = n_1$ and

$$\max_{\tilde{z} \in A^c} \prod_{i=1}^{k} (n_i - z_i) < (n_1 - \gamma) \prod_{i=2}^{k} n_i . \qquad (3.19)$$

<u>Proof.</u> The necessity is immediate from Lemma 3.2. From Lemma 3.3, we then have that $A \subset A^*$. From (3.19), it follows immediately that $A^* \subset A$, establishing the conclusion:

<u>Theorem 3.2.</u> Let $0 < n_1 < n_2 < \cdots < n_k$, $k > 2$, $0 < x_i < n_i$, $i = 1,2,\ldots,k$ be given, where $\sum_{i=1}^{k} x_i < n_1$. Then a sufficient condition that the Buehler optimal lower confidence limit obtained by the method of key test results coincides with that given by the use of the maximum likelihood estimator as an ordering function is

$$\left(\frac{(n_j - \gamma) + \sum_{i=j+1}^{k} n_i - 1}{k - j + 1} \right)^{k-j+1} \prod_{i=1}^{j-1} n_i < (n_1 - \gamma) \prod_{i=2}^{k} n_i , \qquad (3.20)$$

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where j is specified by Lemma 3.4. In this case, the method of key test results is valid for all $x_1 < \gamma$, where γ satisfies (3.20).

Then

$$P_{\underset{p}{\sim}}\{g(\tilde{x}) > g(\tilde{x}_0)\} < I_{\prod_{i=1}^{k} p_i}(n_1 - x_1, x_1 + 1) , \qquad (3.21)$$

where

$$I_p(n - t, t + 1) = \frac{1}{B(n - t, t + 1)} \int_0^p u^{n-t-1}(1 - u)^t du = \sum_{i=0}^{t} \binom{n}{i} p^{n-i} q^i . \qquad (3.22)$$

<u>Proof.</u> The proof is an immediate consequence of Lemmas 3.4 and 3.5 and the following observations. The set $A = \{z : \sum_{i=1}^{k} z_i < \Sigma x_i\}$ is the set whose probability is sought in (1.4), when the method of key test results is utilized and the set $A^* = \{z : \prod_{i=1}^{k} (n_i - z_i) > \prod_{i=1}^{k} (n_i - x_i)\}$ is the corresponding set when $m(x) = g(x)$.

Also,

$$\max_{z \in A^c} \prod_{i=1}^{k} (n_i - z_i) < \left(\frac{(n_j - \gamma) + (\sum_{j+1}^{k} n_i - 1)}{k - j + 1} \right)^{k-j+1} \prod_{i=1}^{j-1} n_i < (n_1 - \gamma) \prod_{i=2}^{k} n_i ,$$

j as specified in (3.12), since the maximum as given in Lemma 3.4 was calculated over a set of real numbers and the application requires that $x_1, x_2, \ldots, x_k$; $z_1, z_2, \ldots, z_k$ be integer valued.

<u>Remark.</u> Some comments are required in applying Lemma 3.4. The solutions are real valued. However, the application requires integer values. From non-integral solutions, utilizing the strict concavity, one can easily determine the required integers.

Lemma 3.5 shows that there is a largest γ for which $A_\beta^* \subset A_\beta$ for all $0 < \beta < \gamma$. If $\gamma = 0$, from the Corollary to Lemma 4.2, we have that $A_\gamma^* \subset A_\gamma$. If $\gamma = n_1 - 1$, then for $A_\gamma^* \quad A_\gamma$, we must have

$$\prod_{i=1}^k (n_i - z_i) < \prod_{i=2}^k n_i < \left(\sum_{i=2}^k n_{i/k}\right)^k .$$

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In order to obtain lower confidence limits for the reliability of series systems using binomial subsystem data, K. A. Weaver introduced the method of "key test results". This work was extended by A. Winterbottom. In the present paper, conditions are obtained under which the "method of key test results" gives Buehler optimal lower confidence limits identical with those given by the ordering induced by the maximum likelihood estimator.

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